Chapter 3: Counting Reference Sheet

Combining Outcomes

We speak of counting the number of **outcomes** that can result from a given **event**. In terms of sets, an outcome is an element from the set that satisfies the requirements defined in the event. So counting outcomes is finding the size of the subset of elements meeting the event requirements.

Two Ways to Think of Combining Outcomes

There are two ways we can think of combining outcomes:

1. We can combine the *sets* of outcomes.
2. We can combine the *outcomes* in the sets.

Sum Principle

If an event A results in m outcomes, and event B results in n *disjoint* outcomes, then the event $A \text{ or } B$ results in $m + n$ outcomes.

Product Principle

If event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event $A \text{ and } B$ can occur in $m \cdot n$ ways.

Additive Principle (with sets)

If event A can occur in m ways, and event B can occur in n *disjoint* ways, then the event A or B can occur in $m + n$ ways.

Stated using set notation, we have:

Given two sets A and B, if $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|$$

Multiplicative Principle

If event A can occur in m ways, and each possibility for A allows exactly n ways for event B, then the event A and B can occur in $m \cdot n$ ways.

Stated using set notation, given two sets A and B we have

$$|A \times B| = |A| \cdot |B|$$

Cartesian Product

Given sets A and B, we can form the set

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}$$

to be the set of all ordered pairs (x, y) where x is an element of A and y is an element of B. We call $A \times B$ the **Cartesian product** of A and B.

Cardinality of a Union of Two Sets

For any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Cardinality of a Union of Three Sets

For any finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

Principle of Inclusion/Exclusion (PIE)

This principle can be applied to any number of sets, but it becomes ever more complicated as the number of sets increases. For four sets, we have:

$$\begin{gathered} |A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| \\ -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ -|A \cap B \cap C \cap D|. \end{gathered}$$

Permutation

A **permutation** is a (possible) rearrangement of objects. For example, the six permutations of the letters a, b, and c are:

> abc, acb, bac, bca, cab, cba

Permutations of n elements

There are

$$n! = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 2 \cdot 1$$

permutations of n (distinct) elements.

$k\text{-permutations}$ of n elements

$P(n,k)$ is the number of $k\text{-permutations}$ **of n elements**, the number of ways to arrange k objects chosen from n distinct objects.

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)...(n-(k-1))$$

> Note that when $n = k$, we have $P(n,n) = \frac{n!}{(n-n)!} = n!$, since we defined $0! = 1$.

Closed formula for $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)...(n-(k-1))}{k(k-1)(k-2)...1}$$

Multisets

A **multiset** is an unordered collection of elements, each of which can appear any number of times. The number of times an element appears is called its **multiplicity**.

Multisets are written using the same noation as sets: a comma-seperated list in braces, such as $\{1,2,2,5\}$.

Probability Definition

Suppose a random experiment has sample space S. The **probability** of an event E is the number of outcomes in E divided by the number of outcomes in S. We write this as $P(E) = \frac{|E|}{|S|}$.

Probability of the Complement Theorem

The probability of the complement of an event E is

$$P(\bar{E}) = 1 - P(E).$$

Probability of Disjoint Events Theorem

Suppose A and B are two **disjoint** events. Then the probability of either A or B happening is,

$$P(A \cup B) = P(A) + P(B).$$

If A and B are not disjoint, then the probability of A or B occuring is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Independent Events Definition

Given two events A and B, we say they are **independent** provided the probability of both events happening is the product of the probabilities of each event happening:

$$P(A \cup B) = P(A) \cdot P(B).$$

Conditional Probability Definition

Given two events A and B, the **conditional probability** of A given B is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$